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Symmetry breaking and axionic strings in the warped deformed conifold

Steven S. Gubser and Igor R. Klebanov

Joseph Henry Laboratories, Princeton University
Princeton, NJ 08544, U.S.A.
E-mail: ssgubser@Princeton.EDU, klebanov@Princeton.EDU

Christopher P. Herzog

Kavli Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106, U.S.A.
E-mail: herzog@kitp.ucsb.EDU

Abstract: We interpret D-strings at the bottom of the warped deformed conifold as axionic strings in the dual cascading $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ gauge theory. The axion is a massless pseudo-scalar glueball which we find in the supergravity fluctuation spectrum and interpret as the Goldstone boson of spontaneously broken U(1) baryon number symmetry. The existence of this massless glueball, anticipated in [1], supports the idea that the cascading gauge theory is on the baryonic branch, i.e. the $\mathrm{U}(1)_B$ global symmetry is broken by expectation values of baryonic operators. We also find a massless scalar glueball, which is a superpartner of the pseudo-scalar. This scalar mode is a mixture of an NS-NS 2-form and a metric perturbation of the warped deformed conifold of a type first considered in [2].

Keywords: Spontaneous Symmetry Breaking, D-branes, AdS-CFT and dS-CFT Correspondence, Supergravity Models.

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1. Introduction

One of the goals of the string formulation of gauge theories is to find a dual string description of confinement [3]. Indeed, a dual approach to conformal theories, the AdS/CFT correspondence [4]–[6], extends to confining backgrounds. Witten [7] initiated such an approach to non-supersymmetric theories, while the authors of [8, 9] found supergravity solutions dual to $\mathcal{N} = 1$ supersymmetric confining theories. However, the UV structure of these confining theories does not correspond to the conventional asymptotically free gauge theory. In [7, 9] the UV theory exists in a dimension higher than 4 and is not field-theoretic. In [8] the gauge theory remains 4-dimensional but has $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ gauge group [10] whose flow exhibits an RG cascade [8, 11]. The fact that the gauge theory remains 4-dimensional in the UV allows for comparison of the beta-function coefficients in this case [12], but neither gauge group is asymptotically free.

While the UV physics of the confining backgrounds in [8, 9] is exotic, it was generally hoped that they are in the same IR universality class as the pure glue $\mathcal{N} = 1$ gauge theory. In this paper we study this question in some detail for the warped deformed conifold of [8], and reach a conclusion that this is not quite the case. An old puzzle guides our investigation: what is the gauge theory interpretation of D1-branes in the deformed conifold background [8]? The interpretation of the fundamental strings placed in the IR region of the metric is clear: they are dual to confining strings, and this duality can be used to extract results on k-string tensions [13] (see also [14]). Like the fundamental strings, the D-strings fall to the bottom of the throat, $\tau = 0$, where they remain tensionful; hence, they cannot be dual to ‘t Hooft loops which must be screened [8].

Two options remain for resolving this puzzle. The first one is that there is an effect that destabilizes the D-strings or removes them from the IR region. For example, D-strings could blow up into a D3-brane wrapped over an S^2 at some finite τ, similar to how F-strings blow up into a D3-brane wrapping an S^2 at some finite azimuthal angle of the S^3. An implication of the latter effect is that a composite state of M F-strings does not have a tension, in agreement with the dual gauge theory [13]. A detailed study of this effect, presented in appendix B, shows that this blow-up does not occur for D-strings. The fact that we have not found any destabilizing effect or blow-up of D-strings leads us to believe that they are stable objects. Therefore, we have to explore this other option and look at their dual interpretation in the gauge theory.

We propose that in the dual gauge theory they are global strings that create a monodromy of a massless axion field.[1] For this explanation to make sense, the IR gauge theory must differ from the pure glue $\mathcal{N} = 1$ theory in that it contains a massless pseudo-scalar bound state (glueball) that plays the role of the axion field. The study of glueball spectra in the background of [8] is complicated due to mixing of various perturbations. Glueball spectra corresponding to a small subset of fluctuations were calculated in [15]–[17], and no massless modes were found. The fact that the massless mode must couple directly to a D-string means that it corresponds to a certain perturbation of the RR 2-form field which also mixes with the RR 4-form field. We turn to the necessary ansatz in sections 2 and 3, and indeed find a massless glueball. This mode should be interpreted as the Goldstone boson of spontaneously broken global U(1) baryon number symmetry. This Goldstone mode was anticipated by Aharony in [1],[2] and its presence supports the claim made in [8, 1] that the cascading gauge theory is on the baryonic branch [18], i.e. certain baryonic operators acquire expectation values. This interpretation of the massless glueball is presented in section 4. The supersymmetric Goldstone mechanism gives rise also to a massless scalar mode or “saxion.” In section 5 the supergravity dual of this mode is identified as a massless glueball coming from a mixture of an NS-NS 2-form and a metric deformation. The ansatz for such perturbations was written down some time ago by Papadopoulos and Tseytlin [2].

Besides being an interesting example of the gauge/gravity duality, the warped deformed conifold background offers interesting possibilities for solving the hierarchy problem [8, 21]. If the background is embedded into a compact CY space with NS-NS and R-R fluxes, then an exponential hierarchy may be created between the UV compactification scale and the IR scale at the bottom of the throat. Models of this type received an additional boost due to a possibility of fixing all moduli proposed in [22], and a subsequent exploration of cosmology in [23]. In interesting recent papers [24], a new role was proposed for various

[1] We are grateful to E. Witten for emphasizing this possibility to us.

[2] I.R.K. acknowledges very useful discussions on this issue with O. Aharony and M. Strassler early in 2001.

(p, q) strings placed in the IR region. Besides being the confining or axionic strings from the point of view of the gauge theory, they may be realizations of cosmic strings. The exponential warping of the background lowers the tension significantly, and makes them plausible cosmic string candidates. Our improved understanding of D-strings in the warped deformed conifold as axionic strings in the gauge theory should therefore have new implications for this cosmological modeling. In section 5 we discuss the Higgs mechanism that occurs upon embedding the warped deformed conifold into a flux compactification, and make a connection with recent papers on Abrikosov-Nielsen-Olesen strings in supergravity [25] (see also earlier work by [26]). In section 7 we find the supergravity background describing D-strings at $\tau = 0$ that are smeared over their transverse directions x^2, x^3 as well as over the S^3. This may be thought of as a non-commutative deformation of the warped deformed conifold. We conclude in section 8.

2. Ansatz for the 2-form perturbation

As a warmup, we first look for our massless pseudo-scalar glueball in the UV asymptotic form of the warped conifold background constructed in [11]. The metric and hence the calculation are simpler than in the full solution of [8], but the qualitative features of our ansatz will be the same.

To begin, consider a D1-brane extended in two of the four dimensions in $\mathbb{R}^{3,1}$. Because the D1-brane carries electric charge under the R-R three-form F_3, it is natural to think that a pseudoscalar a in four dimensions, defined so that $*_4 da = \delta F_3$, experiences monodromy as one loops around the D1-brane world-volume. But it is complicated to write down a full ansatz for the field generated by a D1-brane. A simpler ansatz is $a(t) \sim t$, which is to say $\delta F_3 \sim f_1 dx^1 \wedge dx^2 \wedge dx^3$. The idea is to add whatever else we need to make this a solution to the linearized equations of motion. A solution so obtained would represent a zero-momentum pseudoscalar. In section 3 we will extend our treatment to arbitrary light-like momenta, which is to say arbitrary harmonic $a(t, x^1, x^2, x^3)$.

Using the conventions of [12], the equations of motion include

$$d(e^{-\phi} * H_3) = -g_s^2 F_5 \wedge F_3\,, \qquad d(e^{\phi} * F_3) = F_5 \wedge H_3\,, \qquad F_5 = *F_5\,, \tag{2.1}$$

while the Bianchi identities include

$$dF_3 = 0\,, \qquad dH_3 = 0\,, \qquad dF_5 = H_3 \wedge F_3\,. \tag{2.2}$$

The background solution [11] has

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{h}}(-dt^2 + d\vec{x}^2) + \sqrt{h}(dr^2 + r^2 ds^2_{T^{11}})\,, \\ F_3 &= \frac{M\alpha'}{2}\omega_3\,, \qquad H_3 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2\,, \qquad B_2 = \frac{3g_s M\alpha'}{2} \log \frac{r}{r_*}\,\omega_2\,, \\ F_5 &= (1 + *) B_2 \wedge F_3\,. \end{aligned} \tag{2.3}$$

where ω_2 and ω_3 are defined in appendix A, (A.11) and (A.12), $T^{1,1}$ is the level surface of the conifold, and $h(r)$ is a harmonic function on the conifold.

Hodge duals can be defined through the equations

$$\nu \wedge *\nu = \nu^2 \,\mathrm{vol}\,, \qquad \nu^2 = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \nu_{i_1 \cdots i_p} \nu_{j_1 \cdots j_p}\,, \tag{2.4}$$

and for the background solution,

$$\mathrm{vol} = \frac{\sqrt{h} r^5}{54} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega_2 \wedge \omega_3 \wedge dr\,. \tag{2.5}$$

Some important identities in the following calculations are

$$\omega_3 = g^5 \wedge \omega_2\,, \qquad dg^5 \wedge \omega_2 = 0\,, \qquad dg^5 \wedge dg^5 = -4\omega_2 \wedge \omega_2\,, \tag{2.6}$$

where g^5 is also defined in appendix A.

The perturbation ansatz we adopt is

$$\begin{aligned} \delta H_3 &= 0\,, \\ \delta F_3 &= f_1 dx^1 \wedge dx^2 \wedge dx^3 + f_2 dt \wedge dg^5 + f_2' dt \wedge dr \wedge g^5\,, \\ \delta F_5 &= f_1 dx^1 \wedge dx^2 \wedge dx^3 \wedge B_2 + f_1 g_s h r \log r \; dt \wedge dr \wedge F_3\,. \end{aligned} \tag{2.7}$$

The variations of all other fields, including the metric and the dilaton, vanish. The Bianchi identity for F_3 implies that f_1 is a constant. The second two terms in δF_3 sum to the exact form $-d(f_2 dt \wedge g^5)$. It's easy to verify that

$$\begin{aligned} d\delta F_5 &= H_3 \wedge \delta F_3\,, \\ g_s^{-2} d * \delta H_3 &= -\delta F_5 \wedge F_3 - F_5 \wedge \delta F_3 \\ &= -f_1 dx^1 \wedge dx^2 \wedge dx^3 \wedge B_2 \wedge F_3 - B_2 \wedge F_3 \wedge f_1 dx^1 \wedge dx^2 \wedge dx^3 = 0\,. \end{aligned} \tag{2.8}$$

The remaining linearized equation of motion is

$$d * \delta F_3 = \delta F_5 \wedge H_3\,. \tag{2.9}$$

To show that this can be satisfied, note first that

$$\begin{aligned} * \delta F_3 &= -f_1 \frac{h^2 r^5}{54} dt \wedge \omega_2 \wedge \omega_3 \wedge dr - f_2 \frac{r}{3} dx^1 \wedge dx^2 \wedge dx^3 \wedge dr \wedge g^5 \wedge dg^5 + \\ &\quad + f_2' \frac{r^3}{6} dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega_2 \wedge \omega_2\,, \\ \delta F_5 \wedge H_3 &= f_1 dx^1 \wedge dx^2 \wedge dx^3 \wedge B_2 \wedge H_3\,. \end{aligned} \tag{2.10}$$

Evidently, the left and right hand sides of eq. (2.9) are both proportional to $dx^1 \wedge dx^2 \wedge dx^3 \wedge dr \wedge \omega_2 \wedge \omega_2$. Collecting terms, we find

$$\left[-\partial_r \frac{r^3}{6} \partial_r + \frac{4r}{3} \right] f_2 = \left(\frac{3 g_s M \alpha'}{2} \right)^2 \frac{\log r/r_*}{r} f_1\,, \tag{2.11}$$

and we recall that f_1 is a constant. In principle, eq. (2.11) may be solved for f_2, but because the background is singular at finite r, the solution doesn't mean very much: we need the full deformed conifold solution in order to set boundary conditions in the IR before we integrate out to the UV region.

3. Generalization to the warped deformed conifold

It turns out that there is a simple generalization of the ansatz (2.7) that is consistent for the complete KS background [8]. Relevant formulae for this background are collected in appendix A. The ansatz is

$$\begin{aligned}
\delta H_3 &= 0\,, \\
\delta F_3 &= f_1 *_4 da + f_2(\tau) da \wedge dg^5 + f_2' da \wedge d\tau \wedge g^5\,, \\
\delta F_5 &= (1+*)\delta F_3 \wedge B_2 = f_1 \left(*_4 da - \frac{\epsilon^{4/3}}{6K^2(\tau)} h(\tau) da \wedge d\tau \wedge g^5 \right) \wedge B_2\,,
\end{aligned} \tag{3.1}$$

where now $f_2' = df_2/d\tau$, $h(\tau)$ is given by (A.6), and $K(\tau)$ by (A.5). We have generalized from $a \sim t$ to an arbitrary harmonic function $a(t, x^1, x^2, x^3)$: thus $d *_4 da = 0$.[3] As before, the second two terms in δF_3 sum to the exact form $-d(f_2 da \wedge g^5)$. The equation $d\delta F_3 = 0$ requires that f_1 is constant; we will set $f_1 = 1$ from here on. The equation $d\delta F_5 = H_3 \wedge \delta F_3$ is satisfied for harmonic a due to the identities

$$H_3 \wedge d\tau \wedge g^5 = 0\,, \qquad H_3 \wedge dg^5 = H_3 \wedge (g^1 \wedge g^4 + g^3 \wedge g^2) = 0\,. \tag{3.2}$$

The Hodge duals are computed with the help of

$$\mathrm{vol} = \frac{\epsilon^4}{96} h^{1/2} \sinh^2 \tau dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5\,. \tag{3.3}$$

Using the explicit form of B_2,

$$B_2 = \frac{g_s M \alpha' (\tau \coth \tau - 1)}{2 \sinh \tau} \left[\sinh^2\left(\frac{\tau}{2}\right) g^1 \wedge g^2 + \cosh^2\left(\frac{\tau}{2}\right) g^3 \wedge g^4 \right], \tag{3.4}$$

we can check that the second term in δF_5 is Hodge dual to the first. Acting with an exterior derivative on this second term makes it vanish.

We also have

$$\begin{aligned}
* \, \delta F_3 &= h^2 \sinh^2 \tau \frac{\epsilon^{4/3}}{96} da \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 - \\
&\quad - f_2 \frac{\epsilon^{4/3}}{6K^2} (*_4 da) \wedge d\tau \wedge g^5 \wedge dg^5 + \\
&\quad + f_2' \frac{3\epsilon^{4/3} K^4 \sinh^2 \tau}{8} (*_4 da) \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4
\end{aligned} \tag{3.5}$$

and

$$\delta F_5 \wedge H_3 = \frac{(g_s M \alpha')^2}{4} (f(\tau) k(\tau))' (*_4 da) \wedge d\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4\,, \tag{3.6}$$

where $f(\tau)k(\tau) = (\tau \coth \tau - 1)^2/4$.

[3] The 4-dimensional Hodge dual $*_4$ is calculated with the Minkowski metric, $\mathrm{vol}_4 = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$.

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The second equation in (2.8) may be checked exactly as in the preceding section.[4] From the remaining equation (2.9), for harmonic a, we then get

$$-\frac{d}{d\tau}[K^4 \sinh^2 \tau f_2'] + \frac{8}{9K^2} f_2 = \frac{(g_s M\alpha')^2}{3\epsilon^{4/3}} (\tau \coth \tau - 1) \left(\coth \tau - \frac{\tau}{\sinh^2 \tau} \right) . \tag{3.7}$$

3.1 Solving the differential equation

To solve this equation, let us first note that the homogeneous equation

$$-\frac{d}{d\tau}[K^4 \sinh^2 \tau y'] + \frac{8}{9K^2} y = 0 \tag{3.8}$$

has the two solutions

$$y_1(\tau) = [\sinh(2\tau) - 2\tau]^{1/3} , \qquad y_2(\tau) = [\sinh(2\tau) - 2\tau]^{-2/3} . \tag{3.9}$$

From the theory of second-order ODE's it is known that the general solution of inhomogeneous equations of the form

$$y'' + p(\tau) y' + q(\tau) y = g(\tau) \tag{3.10}$$

is

$$y = c_1 y_1(\tau) + c_2 y_2(\tau) + Y(\tau) . \tag{3.11}$$

A particular solution is

$$Y(\tau) = -y_1(\tau) \int_0^\tau y_2(x) \frac{g(x)}{W(y_1, y_2)(x)} dx + y_2(\tau) \int_0^\tau y_1(x) \frac{g(x)}{W(y_1, y_2)(x)} dx , \tag{3.12}$$

where the wronskian

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' . \tag{3.13}$$

Using this method we find that the solution of (3.7) that is regular both for small and for large τ is

$$\begin{aligned} f_2(\tau) &= c \left[-K h(\tau) \sinh \tau + \frac{1}{K^2 \sinh^2 \tau} \int_0^\tau dx\, h' K^3 \sinh^3 x \right] \\ &= -\frac{2c}{K^2 \sinh^2 \tau} \int_0^\tau dx\, h(x) \sinh^2 x , \end{aligned} \tag{3.14}$$

where $c \sim \epsilon^{4/3}$. This implies that $f_2 \sim \tau$ for small τ, and $f_2 \sim \tau e^{-2\tau/3}$ for large τ.

We check that δF_3 is normalizable by integrating $\sqrt{-G} |\delta F_3|^2$ over τ where G is the ten dimensional metric. Note that

$$\sqrt{-G} |\delta F_3|^2 d^{10} x = \delta F_3 \wedge * \delta F_3 . \tag{3.15}$$

[4] Note that the ability to satisfy this equation with $\delta H_3 = 0$ depends crucially on having the background values of B_2 and F_3, in addition to F_5. There are no similar solutions for backgrounds where only F_5 is present.

From (3.1) and (3.5), we see that we need to check the convergence of three separate τ integrals. These integrals are

$$\begin{aligned}&\int_0^\infty f_1^2 h(\tau)^2 \sinh(\tau)^2 d\tau\,,\\&\int_0^\infty \frac{f_2(\tau)^2}{K^2} d\tau\,,\\&\int_0^\infty (f_2')^2 K^4 \sinh(\tau)^2 d\tau\,.\end{aligned} \tag{3.16}$$

Each of these three integrals is well behaved at small τ and falls off as $e^{-2\tau/3}$ at large τ. So, this is indeed a normalizable zero-mode of the KS background.

It is interesting to compare this calculation with a similar calculation for the RR-charged MN background [9],

$$ds^2_{str} = e^\phi \left[dx_{||}^2 + N(d\rho^2 + e^{2g}d\Omega^2) + \frac{1}{4}\sum_{a=1}^{3}(\omega^a - A^a)^2\right], \tag{3.17}$$

where

$$e^{2\phi(\rho)} = g_s^2 \frac{\sinh(2\rho)}{2e^{g(\rho)}}\,, \qquad e^{2g(\rho)} = \rho\coth(2\rho) - \frac{\rho^2}{\sinh^2(2\rho)} - \frac{1}{4}\,. \tag{3.18}$$

In this case it is not hard to check that $\delta F_3 = *_4 da$ is a solution of the equations of motion, with all other variations set to zero. However, for the MN metric, the norm

$$\int \sqrt{-G}|\delta F_3|^2 d^{10}x \sim \int_0^\infty d\rho e^{2\phi+2g} \tag{3.19}$$

has an exponential divergence at large ρ. The large ρ limit of the MN background is the metric of D5-branes, which is a linear dilaton background. Such a UV boundary condition generally results in a continuous spectrum of glueballs above a certain gap, i.e. their wavefunctions are not normalizable [27, 28], with a linear divergence of the norm. The fact that the divergence in (3.19) is exponential suggests that this mode should be excluded.[5] Indeed, in the Maldacena-Nunez case there is no obvious continuous symmetry breaking that could produce a Goldstone mode.

4. Interpretation of the zero-mode

Clearly, the zero-mode we have found is a rather special phenomenon that is tied to the properties of the KS background. Usually, zero-mass particles exist due to some spontaneously broken symmetry. We would like to argue, following [1], that the zero-mass glueball we are finding is due to the spontaneously broken global U(1) baryon number symmetry.

Recall that in the context of AdS/CFT correspondence, global symmetries in the boundary CFT become gauge symmetries in the dual bulk supergravity description. Consider for a moment the simpler parent of the KS background, i.e. $AdS_5 \times T^{1,1}$ dual to a

[5]We are grateful to the referee for this suggestion.

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superconformal SU(N) $\times$ SU(N) gauge theory [29, 30]. In this simpler theory, the U(1) baryon number symmetry is preserved, and the gauge field A dual to the baryon number current J^μ is identified as $\delta C_4 \sim \omega_3 \wedge A$ [31, 32]. This CFT also possesses a global U$(1)_R$ symmetry, as well as SU(2) $\times$ SU(2).

In the KS background the SU(2) $\times$ SU(2) is preserved, but the U$(1)_R$ symmetry is broken in the UV by the chiral anomaly down to $\mathbb{Z}_{2M}$ [33]. Further spontaneous breaking of this discrete symmetry to $\mathbb{Z}_2$ does not lead to the appearance of a Goldstone mode. The U$(1)_B$ symmetry is not anomalous, and its spontaneous breaking does lead to the appearance of a Goldstone mode which we found above. The form of the δF_5 in (3.1) makes the connection between our zero-mode and U$(1)_B$ evident. Asymptotically, at large τ, there is a component $\sim \omega_3 \wedge da \wedge d\tau$ in δF_5. Thus, in our case, we have $A \sim da$. For the 4-d effective lagrangian, there should be a coupling

$$\frac{1}{f_a}\int d^4x J^\mu \partial_\mu a = -\frac{1}{f_a}\int d^4x\, a(x)(\partial_\mu J^\mu)\,, \tag{4.1}$$

i.e. the Goldstone boson a enters as the parameter of the baryon number transformation. It is important that this transformation does not leave the vacuum invariant! As discussed in [8, 1] the theory is on the baryonic branch: for example, "the last step" of the cascade takes place through giving expectation values to baryonic operators in the SU($2M$)$\times$SU(M) gauge theory coupled to bifundamental fields A_i, B_j, $i,j = 1,2$. From the point of view of the SU($2M$), the number of flavors equals the number of colors. Hence, in addition to mesonic operators $(N_{ij})^\alpha_\beta$, the gauge theory has baryonic operators invariant under the SU($2M$) $\times$ SU(M) gauge symmetry:

$$\begin{aligned}\mathcal{B} &\sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}}(A_1)^{\alpha_1}_1(A_1)^{\alpha_2}_2\cdots(A_1)^{\alpha_M}_M(A_2)^{\alpha_{M+1}}_1(A_2)^{\alpha_{M+2}}_2\cdots(A_2)^{\alpha_{2M}}_M\,,\\ \bar{\mathcal{B}} &\sim \epsilon^{\alpha_1\alpha_2\ldots\alpha_{2M}}(B_1)^1_{\alpha_1}(B_1)^2_{\alpha_2}\cdots(B_1)^M_{\alpha_M}(B_2)^1_{\alpha_{M+1}}(B_2)^2_{\alpha_{M+2}}\cdots(B_2)^M_{\alpha_{2M}}\,.\end{aligned} \tag{4.2}$$

The baryonic operators are also invariant under the SU(2) $\times$ SU(2) global symmetry rotating A_i, B_j. The superpotential has the form

$$W = \lambda(N_{ij})^\alpha_\beta(N_{k\ell})^\beta_\alpha\epsilon^{ik}\epsilon^{j\ell} + X(\det[(N_{ij})^\alpha_\beta] - \mathcal{B}\bar{\mathcal{B}} - \Lambda^{4M}_{2M})\,. \tag{4.3}$$

The supersymmetry-preserving vacua include the baryonic branch:

$$X = 0\,; \quad N = 0\,; \quad \mathcal{B}\bar{\mathcal{B}} = -\Lambda^{4M}_{2M}\,, \tag{4.4}$$

where the SO(4) global symmetry rotating A_i, B_j is unbroken. Since the supergravity background of [8] also has this symmetry, it is natural to identify the dual of this background with the baryonic branch of the cascading theory. The expectation values of the baryonic operators spontaneously break the U(1) baryon number symmetry $A_k \to e^{i\alpha}A_k$, $B_j \to e^{-i\alpha}B_j$. The deformed conifold as described in [8] corresponds to a vacuum where $|\mathcal{B}| = |\bar{\mathcal{B}}| = \Lambda^{2M}_{2M}$, which is invariant under the exchange of the A's with the B's accompanied by charge conjugation in both gauge groups. As noted in [1], the baryonic branch has complex dimension 1, and it can be parametrized by ξ where

$$\mathcal{B} = i\xi\Lambda^{2M}_{2M}\,, \qquad \bar{\mathcal{B}} = \frac{i}{\xi}\Lambda^{2M}_{2M}\,. \tag{4.5}$$

The pseudo-scalar Goldstone mode must correspond to changing ξ by a phase, since this is precisely what a $\mathrm{U}(1)_B$ symmetry transformation does. As usual, the gradient of the pseudo-scalar Goldstone mode $f_a \partial_\mu a$ is created from the vacuum by the action of the axial baryon number current, J_μ (we expect that the scale of the dimensionful 'decay constant' f_a is determined by the baryon expectation values). To summarize, we see that the breaking of the U(1) baryon number symmetry necessitates the presence of a massless pseudo-scalar glueball, which we have found.

By supersymmetry, our pseudoscalar Goldstone mode falls into a massless $\mathcal{N}=1$ chiral multiplet. Hence, there will also be a massless scalar mode or "saxion" and corresponding Weyl fermion or "axino." The saxion must correspond to changing ξ by a positive real factor.

Let us examine more fully the superfield which contains this U(1) baryon number current: it is

$$\mathcal{U} = \mathrm{tr}\, A_i \bar{A}_i - \mathrm{tr}\, B_j \bar{B}_j\,, \tag{4.6}$$

and $J^\mu \sigma^\mu_{\alpha\dot{\alpha}}$ appears as the $\theta^\alpha \bar{\theta}^{\dot{\alpha}}$ term.[6] The lowest component of $\mathcal{U}$ is

$$\mathcal{O} = \mathrm{tr}\, a_i a_i^* - \mathrm{tr}\, b_i b_i^*\,, \tag{4.7}$$

which was identified in the conformal case as dual to the Kaehler mode that introduces a small resolution of the conifold [31]. Again in the conformal case, J^μ is conserved, its dimension is exactly 3, and correspondingly the dimension of the resolution operator $\mathcal{O}$ is exactly 2. The fact that $\mathcal{O}$ is a scalar operator guarantees that its superpartner J^μ is an axial current (see for example Ch. 26 of [34]). Accordingly, when the corresponding baryon number symmetry is broken, the Goldstone mode is a pseudo-scalar, as we have found above.

To gain some intuition for what operator the scalar mode corresponds to, we may substitute an infinitesimal transformation $a_i \to (1+\lambda)a_i$, $b_i \to (1-\lambda)b_i$ with real λ into the action. The potential terms are invariant, but the kinetic terms

$$S = \int d^4x \;\mathrm{tr}\left[|\partial a|^2 + |\partial b|^2\right] \tag{4.8}$$

are not. For this reason we do not find a conserved Noether current. Instead,

$$\delta S \sim \int d^4x\, \lambda\, \mathrm{tr}\left[a_i^* \Box a_i + a_i \Box a_i^* - b_i^* \Box b_i - b_i \Box b_i^*\right]. \tag{4.9}$$

The dimension 4 operator that enters this action, with certain fermionic and auxiliary field terms added, should correspond to the scalar.

To study the saxion and axino more carefully, we should take advantage of the full power of $\mathcal{N}=1$ supersymmetry. The supersymmetric analog of (4.1) is

$$\int d^4x \int d^4\theta\, \mathcal{U}(D^2+\bar{D}^2)S = -\int d^4x \int d^4\,\theta S(D^2+\bar{D}^2)\mathcal{U}\,, \tag{4.10}$$

[6] For simplicity, we have suppressed the e^V factors in (4.6) which are necessary for $\mathcal{U}$ to be a gauge-invariant operator.

where S is a real superfield containing the axion, $\mathcal{U}$ is defined in (4.6), and $D^2 = D^\alpha D_\alpha$ and $\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$ are squares of the usual superderivatives that commute with the global supercharges. Current conservation means $(D^2 + \bar{D}^2)\mathcal{U} = 0$. (In the language of [34], $\mathcal{U}$ is a linear superfield.)

We now investigate which components of $\mathcal{U}$ act as sources for the axion and saxion. We can write

$$(D^2 + \bar{D}^2)S = \Phi + \Phi^\dagger\,, \tag{4.11}$$

where Φ is a chiral superfield in the conventions of Wess and Bagger [35]:

$$\begin{aligned}\Phi(x) \;=\; & \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) + \\ & + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x)\,.\end{aligned} \tag{4.12}$$

In this language, the axion is $\operatorname{Im}\phi(x)$ and the saxion is $\operatorname{Re}\phi(x)$. Thus, the $\theta\bar{\theta}$ component of $\Phi + \Phi^\dagger$ depends on the axion while both the leading and the D-term of $\Phi + \Phi^\dagger$ depend on the saxion.

As in the bosonic case, we see that $\partial_\mu \operatorname{Im}\phi$ couples to J^μ through the $\theta\bar{\theta}$ component of $\mathcal{U}$. In particular

$$\mathcal{U}|_{\theta\bar{\theta}} = \theta^\alpha\bar{\theta}^{\dot{\alpha}}\left[i\sigma_{\alpha\dot{\alpha}}{}^\mu\left(\operatorname{tr} a_i^*\overleftrightarrow{\partial_\mu} a_i - \operatorname{tr} b_i^*\overleftrightarrow{\partial_\mu} b_i\right) + \cdots\right]. \tag{4.13}$$

The ellipses denote bilinears in the fermions. J_μ can be identified as the term inside the parentheses and is exactly what one would expect. It is an axial current because under parity $a_i \to a_i^*$, and $b_i \to b_i^*$.

Meanwhile, the saxion coupling has two pieces: $\Box\operatorname{Re}\phi(x)$ couples to the leading term of $\mathcal{U}$ while $\operatorname{Re}\phi(x)$ couples to the D-term of $\mathcal{U}$. The D-term of $\mathcal{U}$ is easy to reconstruct from the well known D-term of $\Phi^\dagger\Phi$:

$$(\Phi^\dagger\Phi)\Big|_{\theta^2\bar{\theta}^2} = \theta^2\bar{\theta}^2\left(\frac{1}{4}\Box(\phi^*\phi) - (\partial_\mu\phi^*)(\partial^\mu\phi) + \cdots\right), \tag{4.14}$$

where the ellipses indicate missing fermionic and auxiliary fields.

Putting the two pieces of the saxion coupling together, we find that the $\operatorname{Re}\phi$ dependent piece of (4.10) is

$$\int d^4x(\operatorname{Re}\phi)\left(\frac{1}{2}\Box\left(\operatorname{tr} a_i^* a_i - \operatorname{tr} b_i^* b_i\right) - (\operatorname{tr}(\partial_\mu a_i^*)(\partial^\mu a_i) - \operatorname{tr}(\partial_\mu b_i^*)(\partial^\mu b_i)) + \cdots\right), \tag{4.15}$$

where the ellipses again denote fermionic and auxiliary field dependent terms. This integral is in agreement with (4.9).

To summarize, we note that the gauge theory operator corresponding to the pseudoscalar Goldstone mode is

$$\partial_\mu J^\mu \sim \operatorname{Im}\operatorname{tr}\left[a_i^*\Box a_i - b_i^*\Box b_i\right] + \text{fermion bilinears}\,, \tag{4.16}$$

while the operator corresponding to the scalar is

$$\sim \operatorname{Re}\operatorname{tr}\left[a_i^*\Box a_i - b_i^*\Box b_i\right] + \text{fermion bilinears}\,. \tag{4.17}$$

As expected, the two operators combine into a natural complex operator.

5. The scalar zero-mode

The presence of the pseudo-scalar zero mode found in section 3, and the $\mathcal{N} = 1$ supersymmetry, require the existence of a scalar zero-mode. In this section we argue that this zero-mode comes from a metric perturbation that mixes with the NS-NS 2-form potential. Our argument is based on finding such a deformation (Lorentz invariant in the t, x^1, x^2, and x^3 directions) to first order by solving the equations of motion linearized around the warped deformed conifold background. Because we have not explicitly demonstrated through supersymmetry transformations that this mode and the pseudoscalar mode described in section 3 are part of the same multiplet of $d = 4$, $\mathcal{N} = 1$ supersymmetry, our argument can be regarded as less than airtight. Yet we find it quite compelling that a previously unknown deformation exists with the right general properties to be the saxion.

The warped deformed conifold of [8] has a $\mathbb{Z}_2$ symmetry which interchanges (θ_1, ϕ_1) with (θ_2, ϕ_2), i.e. it interchanges the two S^2's. The metric and the F_5 clearly have this symmetry, while the 3-form field strengths F_3 and H_3 actually change sign under this interchange, so the symmetry transformations should be accompanied by a change of sign.[7] In the gauge theory the corresponding symmetry is the interchange of A_1, A_2 with B_1, B_2 accompanied by the charge conjugation, i.e. the interchange of the fundamental and the antifundamental representations, in the $\mathrm{SU}(M+N)$ and the $\mathrm{SU}(N)$ gauge groups.

It is easy to see that the pseudoscalar mode we found breaks this $\mathbb{Z}_2$ symmetry. The operator corresponding to it, (4.16), changes sign when the A's and B's are interchanged. Also, from the form of the perturbations (3.1) we see that δF_3 is even under the interchange of (θ_1, ϕ_1) with (θ_2, ϕ_2), while F_3 is odd; δF_5 is odd while F_5 is even.

Similarly, the scalar (saxion) mode must also break the $\mathbb{Z}_2$ because in the field theory it breaks the symmetry between expectation values of $|\mathcal{B}|$ and of $|\bar{\mathcal{B}}|$ and couples to a $\mathbb{Z}_2$-odd operator (4.17). Turning on the zero-momentum scalar modifies the background, while a zero-momentum Goldstone mode does not. The reason for the difference is that the scalar changes the absolute value of $|\mathcal{B}|$ and $|\bar{\mathcal{B}}|$ while the Goldstone mode affects only the phase.

Deformations of the warped deformed conifold that break this $\mathbb{Z}_2$ were considered by Papadopoulos and Tseytlin in [2]. They wrote down the following metric ansatz:[8]

$$ds_{10}^2 = e^{2p-x}(e^{2A}dx_{||}^2 + du^2) + ds_5^2\,, \tag{5.1}$$

where

$$\begin{aligned} ds_5^2 = {} & (e^{x+g} + a^2 e^{x-g})\left[(e^1)^2 + (e^2)^2\right] + \\ & + e^{x-g}\left[(e^3)^2 + (e^4)^2 - 2a\left(e^{(3}e^{1)} + e^{(4}e^{2)}\right)\right] + e^{-6p-x}(e^5)^2\,, \end{aligned} \tag{5.2}$$

where p, x, g, A, a are functions of u. This metric preserves the SO(4) symmetry, but the $\mathbb{Z}_2$ is in general broken. It is unbroken only if $e^g + a^2 e^{-g} = e^{-g}$, which is a relation obeyed for the KS solution. In that case the coefficient of $(e^1)^2 + (e^2)^2$ is equal to the coefficient of $(e^3)^2 + (e^4)^2$, so that there is symmetry between the two S^2's. On the other hand, for

[7] We are grateful to E. Witten for illuminating discussions about the $\mathbb{Z}_2$ symmetry.

[8] The radial variable τ is related to u through $d\tau = -e^{4p}du$.

the resolved conifold $a = 0$ and the $\mathbb{Z}_2$ is broken [2, 20]. For the more general ansatz of [2], which also involves a deformation of the 2-form fields, the system of equations was quite complicated and no new explicit solution was found.

In this section we aim to do something simpler: look for the scalar zero-mode by perturbing around the KS solution to linear order in the $\mathbb{Z}_2$ breaking perturbation. The necessary perturbation is a mixture of the NS-NS 2-form and the metric:

$$\begin{aligned}
\delta B_2 &= \chi(\tau) dg^5 \,, \qquad \delta H_3 = \chi' d\tau \wedge dg^5 = \chi' d\tau \wedge (g^1 \wedge g^4 + g^3 \wedge g^2) \,, \\
\delta F_3 &= 0 \,, \qquad \delta F_5 = 0 \,, \\
\delta G_{13} &= \delta G_{24} = m(\tau) \,.
\end{aligned} \tag{5.3}$$

To see that these components of the metric break the $\mathbb{Z}_2$ symmetry, we note that

$$(e^1)^2 + (e^2)^2 - (e^3)^2 - (e^4)^2 = g^1 g^3 + g^3 g^1 + g^2 g^4 + g^4 g^2 \,. \tag{5.4}$$

Now let us check consistency of this ansatz. Since $\delta H_3 \wedge F_3 = 0$, the equation

$$d\delta F_5 = \delta H_3 \wedge F_3 \tag{5.5}$$

is satisfied. Since $F_5 \wedge \delta H_3 = 0$, we must have

$$d\delta(*F_3) = 0 \,, \tag{5.6}$$

where the variation comes entirely from the metric. At linear order in $m(\tau)$ we have

$$\begin{aligned}
G^{13} &= G^{24} = -G^{11} G^{33} m(\tau) \,, \\
G^{11} &= G^{22} = \frac{2}{\epsilon^{4/3} K(\tau) \sinh^2(\tau/2) h^{1/2}(\tau)} \,, \\
G^{33} &= G^{44} = \frac{2}{\epsilon^{4/3} K(\tau) \cosh^2(\tau/2) h^{1/2}(\tau)} \,.
\end{aligned} \tag{5.7}$$

Using this, we find

$$\delta(*F_3) = A(\tau) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \wedge dg^5 \,, \tag{5.8}$$

where $A(\tau)$ is a function we don't need to determine. It follows that (5.6) is indeed satisfied.

Now let us use the equation

$$d\delta(*H_3) = 0 \,, \tag{5.9}$$

which follows from the fact that variations of F_3 and F_5 vanish. This equation is equivalent to

$$H^{\tau 14} = H^{\tau 32} = 0 \,. \tag{5.10}$$

Setting $\alpha' = 1$, and using

$$H_{\tau 12} = \frac{g_s M}{2} f' \,, \qquad H_{\tau 34} = \frac{g_s M}{2} k' \,, \tag{5.11}$$

we find that (5.10) implies the constraint

$$\delta H_{\tau 14} = \delta H_{\tau 32} = \chi' = \frac{g_s M m(\tau)}{2} (f' G^{22} + k' G^{33}) \,. \tag{5.12}$$

We will find it convenient to define

$$m(\tau) = h^{1/2} K \sinh(\tau) z(\tau) = 2^{-1/3} \left[\sinh(2\tau) - 2\tau\right]^{1/3} h^{1/2} z(\tau) \,. \tag{5.13}$$

Then (5.12) becomes

$$\chi' = g_s M z(\tau) \left[f' \coth\left(\frac{\tau}{2}\right) + k' \tanh\left(\frac{\tau}{2}\right)\right] = \frac{1}{2} g_s M z(\tau) \frac{\sinh(2\tau) - 2\tau}{\sinh^2 \tau} \,. \tag{5.14}$$

This is equivalent to [2, eq. (5.21)].

Eqs. (5.9) and (5.10) also imply that $\delta(*_6 H_3) = 0$, where $*_6$ is calculated with the 6-d CY metric. And with (5.12) in hand, one may straightforwardly show that $\delta(*_6 F_3) = -\frac{1}{g_s}\delta H_3$, where the left hand side is non-vanishing even though $\delta F_3 = 0$ because the definition of Hodge duals changes when the metric is varied. The upshot is that the variation we are considering preserves the self-duality condition of the 3-forms, $*_6 H_3 = -g_s F_3$.

5.1 10d Einstein equation

It remains to derive the equation for $m(\tau)$ by linearizing the 13 component of the Einstein equations. The Einstein equation is

$$R_{ij} = \frac{g_s^2}{96} F_{iabcd} F_j{}^{abcd} + \frac{1}{4} H_{iab} H_j{}^{ab} - \frac{1}{48} G_{ij} H_{abc} H^{abc} + \frac{g_s^2}{4} F_{iab} F_j{}^{ab} - \frac{g_s^2}{48} G_{ij} F_{abc} F^{abc} \,, \tag{5.15}$$

where we omitted the dilaton and the RR scalar terms because they do not contribute at first order in $m(\tau)$.

We computed δR_{13} with a computer algebra package. The standard formula for calculating first order perturbations to the Ricci curvature is

$$\delta R_{ij} = \frac{1}{2} \left(-\delta G_a{}^a{}_{;ij} - \delta G_{ij;a}{}^a + \delta G_{ai;j}{}^a + \delta G_{aj;i}{}^a \right) .$$

The covariant derivatives are taken and the index raising done with respect to the unperturbed metric. The first term in this expression vanishes because the metric perturbation is traceless. However, the remaining three terms combine to give

$$\begin{aligned}
\delta R_{13} &= -\frac{3}{\epsilon^{4/3}} K^3 \sinh(\tau) z \Bigg[\frac{K''}{K} + \frac{1}{2}\frac{h''}{h} + \frac{z''}{z} + \frac{(K')^2}{K^2} - \frac{1}{2}\frac{(h')^2}{h^2} + \frac{K'}{K}\frac{h'}{h} + 2\frac{K'}{K}\frac{z'}{z} + \\
&\qquad + \coth\tau \left(\frac{h'}{h} + 4\frac{K'}{K} + 2\frac{z'}{z}\right) + 2 - \frac{1}{\sinh(\tau)^2} - \frac{4}{9}\frac{1}{\sinh(\tau)^2 K^6}\Bigg] \\
&= -\frac{3}{\epsilon^{4/3}} K^3 \sinh(\tau) z \Bigg[\frac{1}{2} \frac{\left((K\sinh(\tau))^2 (\ln h)'\right)'}{(K\sinh(\tau))^2} + \frac{\left((K\sinh(\tau))^2 z'\right)'}{(K\sinh(\tau))^2 z} - \frac{2}{\sinh(\tau)^2} - \\
&\qquad - \frac{8}{9}\frac{1}{K^6 \sinh(\tau)^2} + \frac{4}{3}\frac{\cosh(\tau)}{K^3 \sinh(\tau)^2} \Bigg] ,
\end{aligned} \tag{5.16}$$

where $z(\tau)$ is defined as in (5.13). Note that $\delta R_{13} = \delta R_{24}$ are the only nonzero first order perturbations to the Ricci curvature.

Now we need to calculate the necessary terms linear in m on the RHS of (5.15). From the 5-form we find the source

$$\begin{aligned}\frac{g_s^2}{96}F_{1abcd}F_3{}^{abcd} &= -\frac{g_s^2}{4}F_{12345}^2 G^{22}G^{44}G^{31}G^{55} = m(\tau)\frac{g_s^2}{4}F_{12345}^2 G^{11}G^{22}G^{33}G^{44}G^{55}\\ &= m(\tau)S(\tau)\,,\end{aligned}\tag{5.17}$$

where

$$S(\tau) = \frac{3(g_sM)^4}{2^{1/3}\epsilon^{20/3}h^{5/2}}\frac{(\tau\coth\tau-1)^2[\sinh(2\tau)-2\tau]^{4/3}}{\sinh^6(\tau)}\,.$$

The above can also be written as $z\tilde{S}$ where

$$\begin{aligned}\tilde{S} &= \frac{3}{2^{2/3}}\frac{(g_sM)^4}{\epsilon^{20/3}h^2}\frac{(\tau\coth\tau-1)^2[\sinh(2\tau)-2\tau]^{5/3}}{\sinh^6(\tau)}\\ &= \frac{3}{2\cdot\epsilon^{4/3}}K^3\sinh\tau\frac{(h')^2}{h^2}\,.\end{aligned}\tag{5.18}$$

From the incompletely contracted RR and NSNS 3-form terms we get

$$\begin{aligned}\frac{g_s^2}{4}F_{iab}F_j{}^{ab} &= \frac{g_s^2}{2}\left[F_{125}F_{345}G^{24}G^{55} + F_{13\tau}F_{31\tau}G^{13}G^{\tau\tau}\right]\\ &= m(\tau)\frac{(g_sM)^2}{8}\left[-F(1-F)G^{22}G^{44}G^{55} + (F')^2G^{11}G^{33}G^{\tau\tau}\right]\,,\end{aligned}\tag{5.19}$$

and

$$\begin{aligned}\frac{1}{4}H_{1ab}H_3{}^{ab} &= \frac{1}{2}\Bigg[H_{12\tau}H_{34\tau}G^{24}G^{\tau\tau} + H_{135}H_{315}G^{13}G^{55} + H_{14\tau}H_{34\tau}G^{44}G^{\tau\tau} +\\ &\qquad + H_{12\tau}H_{32\tau}G^{22}G^{\tau\tau}\Bigg]\\ &= m(\tau)\frac{(g_sM)^2}{8}G^{55}\left[-f'k'G^{22}G^{44} + (F')^2G^{11}G^{33} + (f'G^{22}+k'G^{33})^2\right]\\ &= m(\tau)\frac{(g_sM)^2}{8}G^{55}\Bigg[(F')^2G^{11}G^{33} + F(1-F)G^{22}G^{44} + (1-F)^2G^{33}G^{44} +\\ &\qquad + F^2G^{11}G^{22}\Bigg]\,.\end{aligned}\tag{5.20}$$

In the last step we used the identities $f' = (1-F)\tanh^2(\tau/2)$ and $k' = F\coth^2(\tau/2)$. For the remaining contribution of the 3-forms, we use the fact that for the unperturbed solution $g_s^2|F_3|^2 = |H_3|^2$ and only compute $|F_3|^2$:

$$\begin{aligned}-\frac{g_s^2}{24}G_{13}F_{abc}F^{abc} &= -m\frac{(g_sM)^2}{8}\Bigg[(F')^2G^{11}G^{33}G^{\tau\tau} + \frac{1}{2}F^2G^{11}G^{22}G^{55} +\\ &\qquad + \frac{1}{2}(1-F)^2G^{33}G^{44}G^{55}\Bigg]\,.\end{aligned}\tag{5.21}$$

We see that there are some cancellations, and the contributions from 3-form sources (5.19)–(5.21) combine into

$$\begin{aligned}m(\tau)\frac{g_s^2}{24}F_{abc}F^{abc} &= m(\tau)\frac{(g_sM)^2}{8}G^{55}\left[(F')^2G^{11}G^{33} + \frac{1}{2}(1-F)^2G^{33}G^{44} + \frac{1}{2}F^2G^{11}G^{22}\right]\\ &= m(\tau)T(\tau)\end{aligned}\tag{5.22}$$

where

$$T(\tau) = \frac{3(g_s M)^2}{8\epsilon^4 h^{3/2}} \frac{\cosh(4\tau) + 8(1+\tau^2)\cosh(2\tau) - 24\tau\sinh(2\tau) + 16\tau^2 - 9}{\sinh^6(\tau)} . \tag{5.23}$$

The above can also be written as $z\tilde{T}$ where

$$\begin{aligned}
\tilde{T} &= \frac{3}{8\cdot 2^{1/3}} \frac{(g_s M)^2}{\epsilon^4 h} \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{\sinh^6(\tau)} \times \\
&\quad \times \big(\cosh(4\tau) + 8(1+\tau^2)\cosh(2\tau) - 24\tau\sinh(2\tau) + 16\tau^2 - 9\big) \\
&= -\frac{3}{\epsilon^{4/3}} K^3 \sinh\tau \left[\frac{1}{2}\frac{h''}{h} + \frac{K'}{K}\frac{h'}{h} + \coth\tau\frac{h'}{h}\right] .
\end{aligned} \tag{5.24}$$

Putting the pieces together, we can write the Einstein equation (5.15) as

$$\delta R_{13} = (\tilde{S} + \tilde{T}) z(\tau) .$$

Remarkably, this differential equation for $z(\tau)$ is independent of h. The $\tilde{S}$ exactly cancels the $1/h^2$ dependent piece of δR_{13} and the $\tilde{T}$ precisely cancels the $1/h$ dependent terms. The remaining differential equation for $z(\tau)$ is precisely (5.16) set to zero with h set to a constant:

$$\frac{\left((K\sinh(\tau))^2 z'\right)'}{(K\sinh(\tau))^2} = \left(\frac{2}{\sinh(\tau)^2} + \frac{8}{9}\frac{1}{K^6\sinh(\tau)^2} - \frac{4}{3}\frac{\cosh(\tau)}{K^3\sinh(\tau)^2}\right) z . \tag{5.25}$$

To gain confidence in this differential equation for $z(\tau)$, we rederived it using the 1d effective action of PT [2]. In particular, the metric functions in (5.2) can be reexpressed in terms of z:

$$e^g = \frac{1}{\cosh y - cz}, \qquad a = \frac{\sinh y}{\cosh y - cz} , \tag{5.26}$$

where c is a numerical constant. With this ansatz, one can write down a 1d effective action for $z(\tau)$, keeping only terms of order z^2. The lagrangian is

$$\begin{aligned}
L &= \frac{z(\tau)^2\left(-4 + 6\,K(\tau)^3\,\sinh(\tau)\,\tanh(\tau) + 9\,K(\tau)^5\,\tanh(\tau)\,K'(\tau)\right)}{9\,K(\tau)^4} - \\
&\quad -\frac{1}{2}K(\tau)^2\sinh(\tau)^2\, z'(\tau)^2 .
\end{aligned} \tag{5.27}$$

The corresponding equation of motion is precisely (5.25). Instead of presenting a derivation of (5.27) here, we have made it available online [19].

5.2 Analysing the scalar mode

The solution of (5.25) for the zero-mode is remarkably simple:

$$z(\tau) = \frac{c_1\coth(\tau) + c_2(\tau\coth(\tau) - 1)}{[\sinh(2\tau) - 2\tau]^{1/3}} , \tag{5.28}$$

with c_1 and c_2 constants. We must set $c_1 = 0$ in order for the zero-mode to be well behaved at $\tau = 0$. Like the pseudoscalar perturbation, the large τ asymptotic is again $z \sim \tau e^{-2\tau/3}$. We note that the metric perturbation also has the simple form $\delta G_{13} \sim h^{1/2}[\tau \coth(\tau) - 1]$. Note also that the perturbed metric $d\tilde{s}_2^6$ differs from the metric of the deformed conifold by

$$\sim (\tau \coth \tau - 1)(g^1 g^3 + g^3 g^1 + g^2 g^4 + g^4 g^2) \,, \tag{5.29}$$

which grows as $\ln r$ in the asymptotic radial variable r.

The existence of the scalar zero-mode makes it likely that there is a one-parameter family of supersymmetric solutions to the Papadopoulos-Tseytlin equations [2], obtained from their ansatz (5.2), which break the $\mathbb{Z}_2$ symmetry interchanging the two S^2's. We will call these conjectured backgrounds *resolved warped deformed conifolds*. We add the word *resolved* because both the resolution of the conifold, which is a Kaehler deformation, and these resolved warped deformed conifolds break the $\mathbb{Z}_2$ symmetry. (Note that at the special point where the $\mathbb{Z}_2$ breaking parameter vanishes, the resolved warped deformed conifold becomes an ordinary warped deformed conifold.)

As we explained in section 4, in the dual gauge theory turning on the $\mathbb{Z}_2$ breaking corresponds to the transformation $\mathcal{B} \to \xi \mathcal{B}$, $\bar{\mathcal{B}} \to \xi^{-1} \bar{\mathcal{B}}$ on the baryonic branch. Therefore, ξ is dual to the $\mathbf{Z_2}$ breaking parameter of the warped deformed conifold.

One might ask whether our resolved warped deformed conifolds are still of the form $h^{-1/2} dx_{||}^2 + h^{1/2} d\tilde{s}_6^2$ where $d\tilde{s}_6^2$ is Ricci flat. At linear order in our perturbation, our conifold metric $d\tilde{s}_6^2$ is indeed Ricci flat: the first order corrections vanish if (5.25) is satisfied. We also showed (see below (5.14)) that the complex 3-form field strength $G_3 = F_3 - \frac{i}{g_s} H_3$ remains imaginary self-dual at linear order, i.e. $*_6 G_3 = i G_3$. It will be interesting to see if these properties continue to hold for the exact solution.

6. Compactification and Higgs mechanism

As we argued above, the non-compact warped deformed conifold exhibits a supergravity dual of the Goldstone mechanism. It was crucial for our arguments that the $\mathrm{U}(1)_B$ symmetry is not gauged in the field theory, and the appearance of the Goldstone boson in the supergravity dual confirms the symmetry is global.

If the warped deformed conifold is embedded into a flux compactification of type IIB string on a 6-dimensional CY manifold, then we expect the global $\mathrm{U}(1)_B$ symmetry to become gauged, because the square of the gauge coupling becomes finite. In the compact case we may write $\delta C_4 \sim \omega_3 \wedge A$, where ω_3 is harmonic in the full compact case and A is the 4-d gauge field. If we ignore subtleties with the self-duality of the 5-form field strength, then the kinetic terms for it is

$$\frac{1}{2g_s^2} \int d^{10}x \sqrt{-g} F_5^2 \,. \tag{6.1}$$

Substituting $F_5 = F_2 \wedge \omega_3$ and reducing to 4 dimensions, we find the U(1) kinetic term

$$\frac{1}{2g^2} \int d^4x F_2^2 \,, \tag{6.2}$$

where

$$\frac{1}{g^2} \sim \frac{1}{g_s^2}\tau_m\,, \tag{6.3}$$

where we assumed that the effect of compactification is to introduce a cut-off at $\tau_m \gg 1$.

The finiteness of the gauge coupling in the compact case means that the Goldstone mechanism should turn into a Higgs mechanism. The Goldstone boson a enters as a gauge parameter of A and gets absorbed by the U(1) gauge field to make a massive vector field. As usual in the supersymmetric Higgs mechanism, the scalar acquires the same mass. As a result, we find an $\mathcal{N} = 1$ massive vector supermultiplet containing a massive vector, a scalar (the Higgs boson), and their fermion superpartners.

While in the non-compact case D-strings are global strings, in the compact case they should be interpreted as Abrikosov-Nielsen-Olesen vortices of an abelian-Higgs model, where the charged chiral superfields breaking the gauge symmetry are the baryon operators $\mathcal{B}$ and $\bar{\mathcal{B}}$. Representation of D-strings by ANO vortices in low-energy supergravity was recently advocated in [25], where it was argued that such vortices can be BPS saturated if an all-important Fayet-Iliopoulos D-term is included. Note, however, that D1-branes placed at the bottom of the throat embedded into a flux compactification cannot be BPS. Since there is a finite number K of NS-NS flux units through a cycle dual to the 3-sphere [21], the D-string charge takes values in $\mathbb{Z}_K$: now K D-strings can break on a wrapped D3-brane [24]. Correspondingly, we do not expect the ANO vortex duals to be BPS saturated. We hope to return to investigation of these issues in future work.

7. A double T-duality on the KS solution

In this section we find the supergravity background describing D-strings at $\tau = 0$ that are smeared over their transverse directions x^2, x^3 as well as over the S^3. The resulting solution has B_{23} turned on, and may be thought of as a non-commutative deformation of the warped deformed conifold solution. A strategy that was useful in obtaining the non-commutative generalization of $AdS_5 \times S^5$ was to use double T-duality along the x^2, x^3 directions [36, 37].

We will use the rules for T-duality from page 30 of the paper of Bergshoeff, Hull, and Ortin (BHO) [38]. We want to T-dualize the KS supergravity solution first in the z direction and then in the $z\cos\theta - y\sin\theta$ direction where our gauge theory coordinates are x, y, z, and t. The BHO rules simplify quite a bit because to start with there is no NS-NS B_2 or RR C_2 potential in the z direction. Moreover, the metric is diagonal, and the axion field is turned off.

Let the hatted fields be the result of the two T-dualities. Two new coordinates are needed at the end of the day:

$$y' = y\cos\theta + z\sin\theta \tag{7.1}$$

and the T-dual z' of $z\cos\theta - y\sin\theta$. Recall that the BHO rules are worked out in the string frame.

The transformation rules for the dilaton, metric, and NS-NS B field can be found in many places in addition to the BHO paper. After two T-dualities, the string frame metric becomes

$$ds_{10}^2 = h(\tau)^{-1/2}(-dt^2 + dx^2 + \chi(\tau)(dy^2 + dz^2)) + h(\tau)^{1/2} ds_6^2 \,, \tag{7.2}$$

where ds_6^2 is the usual warped deformed conifold metric (A.2) and

$$\chi(\tau)^{-1} = \cos^2\theta + h(\tau)^{-1}\sin^2\theta \tag{7.3}$$

can be written in terms of the usual warp factor $h(\tau)$. The dilaton becomes

$$e^{2\hat{\phi}} = g_s^2 \chi(\tau) \,. \tag{7.4}$$

(We have departed from the convention of the previous sections where e^{ϕ} was independent of g_s.) The NS-NS two form in the conifold directions is unchanged:

$$\hat{B}_{\alpha\beta} = B_{\alpha\beta} \,. \tag{7.5}$$

At the same time, $\hat{B}_2$ picks up a $y'z'$ dependent piece:

$$\hat{B}_{y'z'} = \tan\theta(1 - h^{-1}\chi) \,. \tag{7.6}$$

This $y'z'$ dependent piece becomes more familiar if we add a constant $\tan\theta$:

$$\hat{B}_{y'z'} - \tan\theta = -h^{-1}\chi\tan\theta \tag{7.7}$$

which matches (2.1) of the Maldacena-Russo paper [37] up to a sign. Our sign conventions appear to be necessary to cancel the minus sign in $F_5 = dC_4 - C_2 \wedge H_3$.

T-duality transformation rules for the RR potentials are less well known, and we really rely on the BHO paper here. Unfortunately, our conventions for the SUGRA fields are rather different from the BHO paper. We have fixed signs and overall normalizations by checking these transformation rules using the SUGRA equations of motion. For the four form and two form potentials in the gauge theory directions,

$$\hat{C}_{txy'z'} = \chi C_{txyz}\cos\theta \,, \tag{7.8}$$

and

$$\hat{C}_{tx} = C_{txyz}\sin\theta \tag{7.9}$$

which agrees with (2.1) of the Maldacena-Russo paper [37]. To show agreement, we have used the fact that in these types of warped compactifications, $g_s C_{txyz} = h^{-1}$ where h is the warp factor and also that $F_5 = dC_4 - C_2 \wedge H_3$. As in [37], we find

$$g_s\hat{C}_{tx} = h^{-1}\sin\theta \,, \qquad g_s\hat{F}_{txy'z'\tau} = \chi\partial_\tau h^{-1}\cos\theta \,. \tag{7.10}$$

In the KS solution, we also have to trace how F_3 transforms:

$$\hat{C}_{\alpha\beta y'z'} = -C_{\alpha\beta}h^{-1}\chi\sin\theta \,, \tag{7.11}$$

$$\hat{C}_{\alpha\beta} = C_{\alpha\beta}\cos\theta \,. \tag{7.12}$$

These rules are reminiscent of the results (2.9)–(2.12) of Mateos, Pons, and Talavera [39]. However, their T-duality prescription is not completely the same as ours. The minus sign in $\hat{C}_{\alpha\beta y'z'}$ is necessary to cancel a C_2 dependent piece in F_5. From the equation for $\hat{C}_{\alpha\beta y'z'}$, we have

$$\hat{F}_{tx\alpha\beta\gamma} = -\hat{C}_{tx} H_{\alpha\beta\gamma} \,, \tag{7.13}$$

and the component

$$\hat{F}_{y'z'\alpha\beta\gamma} = -h^{-1} \chi \sin\theta F_{\alpha\beta\gamma} \tag{7.14}$$

related to it by Hodge duality.

We have checked that the hatted fields satisfy the string frame dilaton equation of motion

$$d * de^{-2\hat{\phi}} = e^{-2\hat{\phi}} \hat{H}_3 \wedge *\hat{H}_3 - \hat{F}_3 \wedge *\hat{F}_3 \tag{7.15}$$

and the equation of motion for $\hat{F}_3$:

$$d * \hat{F}_3 = \hat{F}_5 \wedge \hat{H}_3 \,. \tag{7.16}$$

8. Discussion

Our paper sheds new light on the physics of the cascading $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ gauge theory, whose supergravity dual is the warped deformed conifold [8]. In the infrared the theory is not in the same universality class as the pure glue $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(M)$ theory: the cascading theory contains a massless pseudoscalar glueball, as well as global strings dual to the D-strings placed at $\tau = 0$ in the supergravity background.

As suggested in [8, 1] and reviewed in section 4 above, the infrared field theory is better thought of as $\mathrm{SU}(2M) \times \mathrm{SU}(M)$ on the baryonic branch, i.e. with baryon operators (4.2) having expectation values. Since the global baryon number symmetry, $\mathrm{U}(1)_B$, is broken by these expectation values, the spectrum must contain Goldstone bosons. However, these modes were not found in the spectrum of supergravity fluctuations until now. In this paper we find explicitly the pseudo-scalar Goldstone mode, and we construct at linear order a Lorentz-invariant deformation of the background which we argue is a zero-momentum state of the scalar superpartner of the Goldstone mode. These calculations confirm the validity of the baryonic branch interpretation of the gauge theory. This resolves a puzzle about the dual of the D-strings at $\tau = 0$: they are the solitonic strings that couple to these massless glueballs. We further argue that, upon embedding this theory in a warped compactification, the global $\mathrm{U}(1)_B$ symmetry becomes gauged; then the gauge symmetry is broken by the baryon expectation values through a supersymmetric version of the Higgs mechanism. Thus, in a flux compactification, we expect the D-string to be dual to an Abrikosov-Nielsen-Olesen vortex.

In [8] it was argued that there is a limit, $g_s M \to 0$,[9] where the physics of the cascading gauge theory should approach that of the pure glue $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(M)$ gauge theory. The reader may wonder how this statement can be consistent with the presence

[9]No string theoretic description of this limit is yet available, because it is the opposite of the limit of large $g_s M$ where the supergravity dscription is valid.

of the Goldstone bosons. We believe that it can. Returning to the SU($2M$) $\times$ SU(M) gauge theory discussed in section 4, we expect that in the limit $g_s M \to 0$ the scale Λ_{2M} of the SU($2M$), i.e. that of the baryon condensates, is much higher than the scale Λ_M of the SU(M). Hence, the decay constant f_a should be much greater than the confinement scale Λ_M. Since the Goldstone boson interactions at the confinement scale are suppressed by powers of Λ_M/f_a, they appear to decouple from the massive glueballs containing the physics of the pure glue supersymmetric SU(M) gauge theory. Obviously, this heuristic argument needs to be subjected to various checks.

Our work opens new directions for future research. Turning on finite scalar perturbations is expected to give rise to a new class of Lorentz invariant supersymmetric backgrounds, *the resolved warped deformed conifolds*, which preserve the SO(4) global symmetry but break the $\mathbb{Z}_2$ symmetry of the warped deformed conifold. The ansatz for such backgrounds was proposed in [2]. We have argued that these conjectured backgrounds are dual to the cascading gauge theory on the baryonic branch. It would be desirable to find them explicitly, and to confirm their supersymmetry.

A more explicit construction of the global string as a soliton in the gauge theory is desirable. It is also interesting to explore the consequences of our results for cosmological modeling.

We hope that future work leads to new insights into the remarkable physics of cascading gauge theories.

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A. Formulae for the KS solution

Here we collect some necessary formulae from [8] (for reviews see [12]).

The ten dimensional metric for the KS solution is

$$ds_{10}^2 = h(\tau)^{-1/2}(-dt^2 + dx^2 + dy^2 + dz^2) + h(\tau)^{1/2} ds_6^2 \,, \tag{A.1}$$

where

$$ds_6^2 = \frac{\epsilon^{4/3} K(\tau)}{2} \left[\frac{1}{3K^3}(d\tau^2 + (g_5)^2) + \cosh^2\left(\frac{\tau}{2}\right)((g^3)^2 + (g^4)^2) + \sinh^2\left(\frac{\tau}{2}\right)((g^1)^2 + (g^2)^2) \right] \tag{A.2}$$

is the usual warped deformed conifold metric. The one forms are given in terms of angular coordinates as

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}\,, \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,, \\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}\,, \qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}}\,, \\ g^5 &= e^5\,, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} e^1 &\equiv -\sin\theta_1 d\phi_1\,, \qquad e^2 \equiv d\theta_1\,, \\ e^3 &\equiv \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\,, \\ e^4 &\equiv \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,, \\ e^5 &\equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\,. \end{aligned} \tag{A.4}$$

Note that

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}\,. \tag{A.5}$$

The warp factor is

$$h(\tau) = (g_s M\alpha')^2 2^{2/3}\varepsilon^{-8/3} I(\tau)\,, \tag{A.6}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}\,. \tag{A.7}$$

The NS-NS two form field is

$$B_2 = \frac{g_s M\alpha'}{2}\left[f(\tau) g^1\wedge g^2 + k(\tau) g^3\wedge g^4\right]\,, \tag{A.8}$$

$$H_3 = dB_2 = \frac{g_s M\alpha'}{2}\left[d\tau\wedge(f' g^1\wedge g^2 + k' g^3\wedge g^4) + \frac{1}{2}(k-f) g^5\wedge(g^1\wedge g^3 + g^2\wedge g^4)\right]\,,$$

while the RR three form field strength is

$$\begin{aligned} F_3 &= \frac{M\alpha'}{2}\left\{g^5\wedge g^3\wedge g^4 + d[F(\tau)(g^1\wedge g^3 + g^2\wedge g^4)]\right\} \\ &= \frac{M\alpha'}{2}\left\{g^5\wedge g^3\wedge g^4(1-F) + g^5\wedge g^1\wedge g^2 F + F' d\tau\wedge(g^1\wedge g^3 + g^2\wedge g^4)\right\}\,. \end{aligned} \tag{A.9}$$

The auxiliary functions in these forms are

$$\begin{aligned} F(\tau) &= \frac{\sinh\tau - \tau}{2\sinh\tau}\,, \\ f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1)\,, \\ k(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1)\,. \end{aligned} \tag{A.10}$$

Two harmonic forms important in section 2 are

$$\omega_2 = \frac{1}{2}(g^1\wedge g^2 + g^3\wedge g^4) = \frac{1}{2}(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2)\,, \tag{A.11}$$

$$\omega_3 = g^5\wedge\omega_2\,. \tag{A.12}$$

B. The D-string: blow up or dud?

In [13] it was shown that, under the influence of RR-flux through the S^3, F-strings blow up into a D3-brane wrapped over an S^2 at a finite azimuthal angle within the S^3 at $\tau = 0$. Here we explore the possibility that the NS-NS field within the R^3 fiber makes D1-branes blow up into an S^2 at some small τ.

We wrap the D3-brane in such a way that the pull back of g^1 and g^2 is non-vanishing, while that of g^3, g^4, and g^5 vanishes. In particular, we choose standard angular coordinates (θ, ϕ) on an S^2 such that $\phi = \phi_1 = -\phi_2$, $\theta = \theta_1 = -\theta_2$, and $\psi = 0$. For this choice it follows from (A.3) and (A.4) that the pull back $g^1|_{S^2} = -\sqrt{2}\sin\theta d\phi$ and $g^2|_{S^2} = \sqrt{2} d\theta$.

The tension of the wrapped D3-brane is

$$L = T_3 \sqrt{\det(G + B_2 - (2\pi\alpha')F_2)} \,. \tag{B.1}$$

The gauge field

$$F_2 = \frac{q}{2} \sin\theta d\theta \wedge d\phi$$

is quantized,

$$\int F_2 = 2\pi q \,,$$

where q is the number of D1-branes. The NS-NS two form (A.8) restricted to the two sphere becomes

$$B_2|_{S^2} = g_s M \alpha' f(\tau) \sin\theta d\theta \wedge d\phi \,. \tag{B.2}$$

The induced metric G on the D3-brane can be calculated from (A.1) and is

$$ds^2_{\text{ind}} = h(\tau)^{-1/2}(-dt^2 + dx^2) + h(\tau)^{1/2}\epsilon^{4/3} K \sinh\left(\frac{\tau}{2}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{B.3}$$

Substituting the metric, B_2, and F_2 into the tension, we find that (B.1) becomes

$$L = T_3 \epsilon^{4/3} \left[K^2 \sinh\left(\frac{\tau}{2}\right)^4 + \left(f - \frac{x}{2}\right)^2 2^{-2/3} I(\tau)^{-1} \right]^{1/2} \,, \tag{B.4}$$

where $I(\tau)$ is given by (A.7) and $x = 2\pi q / g_s M$.

We want to analyze this tension to see if there is a minimum for small but non-vanishing τ. As a first step, we expand $I(\tau)$ in a power series near $\tau = 0$:

$$I(\tau) = a_0 - \frac{6^{-1/3}}{3}\tau^2 + \mathcal{O}(\tau^4) \,,$$

and $a_0 = 0.71805\ldots$. We assume that x is also small and write L as a power series in x and τ:

$$L = T_3 \epsilon^{4/3} 2^{-1/3} \left[\frac{x^2}{4a_0} + \frac{x^2\tau^2}{12 \cdot 6^{1/3} a_0^2} - \frac{x\tau^3}{12 a_0} + \frac{\tau^4}{4 \cdot 6^{2/3}} + \cdots \right]^{1/2} \,. \tag{B.5}$$

Clearly, $\tau = 0$ is a minimum of this function.

To minimize L at small but nonzero τ and x, we would need to solve the following quadratic equation

$$\frac{1}{\tau}\frac{dL}{d\tau} \sim \left(\frac{6^{1/3}a_0\tau}{x}\right)^2 - \frac{3}{2}\left(\frac{6^{1/3}a_0\tau}{x}\right) + 1 = 0\,. \tag{B.6}$$

The discriminant of this quadratic equation is negative and there is no minimum for real and nonzero τ; the only minimum is at $\tau = 0$.

Plotting the function (B.4) for a finite x, we see that, again, the only minimum is at $\tau = 0$. Therefore, the blow-up of D1-branes into a wrapped D3-brane does not occur. This should be contrasted with the fundamental strings which do blow up into a D3-brane wrapped over an S^2 within the S^3 at $\tau = 0$ [13]. As a result of this effect, the fundamental string charge takes values in $\mathbb{Z}_M$, and the fundamental strings are not BPS saturated objects.

The D-string charges take values in $\mathbb{Z}$. Also, a D-string at $\tau = 0$ is the object with the smallest tension for a given charge. These facts suggest that the D-strings at $\tau = 0$ are BPS saturated. Consider, however, the T-dual of the conifold with M D5-branes wrapped over the S^2 at the tip. This is provided by the type IIA theory with an NS5-brane spanning the 12345 directions, separated from an NS5-brane spanning the 12367 directions by some distance along the compact x^9 direction. Under the T-duality, the M wrapped D5-branes turn into M D4-branes spanning the 1239 directions and suspended between the NS5-branes [40]. A D-string of type IIB theory turns into a D2-brane of type IIA wrapped over the compact direction, spanning for example the 19 directions. The fact that this object is parallel to the D4-branes appears to break the supersymmetry.

Upon lifting to M-theory the NS5-branes and D4-branes deform and merge into an M5-brane [41]–[43]. This M-theory configuration has two compact coordinates, x^9 and x^{10}. An M2-brane wrapping the x^{10} direction is the dual of the F-string on the conifold, while an M2-brane wrapping the x^9 direction is the dual of the D-string. From this point of view it is less clear what causes the violation of the BPS condition. In any case, it would be interesting to determine whether the D-strings at $\tau = 0$ are BPS through a direct calculation with the warped deformed conifold background of [8].

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